DESIGN OF A TRANSVERSELY LAYERED ROD OF MINIMUM WEIGHT WITH STABILITY CONSTRAINT

V. V. Alekhin and L. V. Baev

UDC 539.3

We consider a problem of synthesis of a transversely layered, axially compressed straight rod of minimum weight from a finite set of elastic homogeneous materials with specified constraint imposed on the critical buckling load. To describe the bending of the rod, we use the classical theory of beams based on the hypothesis of plane cross sections. The necessary optimality conditions are obtained, a computational algorithm is developed, and an example of calculation of the optimal rod is given.

1. Formulation of the Problem. Let W be the set of k homogeneous isotropic materials. The problem is to synthesize a transversely layered rod of minimum weight with constrained critical buckling load applied to the rod.

We consider a rod of length L and constant cross section S, which is compressed by an axial force $\mathbf{P}$. Various types of boundary conditions such as hinged support, fixed ends, and others can be specified at the ends of the rod. We introduce the Cartesian coordinate system (x, y, z) with origin at the left end of the rod and the x axis directed along the rod, so that it coincides with the line of action of the force $\mathbf{P}$ (Fig. 1). The rod buckles in the (x, z) plane.

Let σ and ρ_* be the parameters having the dimensions of stress and density. We introduce the following dimensionless variables (in what follows, the asterisk at the dimensionless quantities is omitted):

$$x^* = x/L, \quad w^* = w/L, \quad E^* = E/\sigma, \quad P^* = PL^2/\sigma I, \quad \rho^* = \rho/\rho_*, \tag{1.1}$$

where $w(x)$ is the deflection of the rod, $E(x)$ and $\rho(x)$ are Young's modulus and the density of the layer material, I is the moment of inertia of the rod cross section S, and P is the magnitude of the force $\mathbf{P}$.

In variables (1.1), the stability equation of the rod has the form [1]

$$\left(Ew''\right)'' + Pw'' = 0 \tag{1.2}$$

(the prime denotes differentiation with respect to the x coordinate). For clarity, we take the boundary conditions

$$w'(0) = (Ew'')'(0) = w(1) = Ew''(1) = 0, \tag{1.3}$$

which give the eigenfunction describing the deflection $w(x)$ of a simply supported rod of double length for the case of a symmetrical buckling shape.

At the internal boundaries $x_i \in (0,1)$ between the layers of the rod, where Young's moduli of the materials have discontinuities, the coupling conditions are to be specified: the continuity of deflection w, slope w', bending moment $M = -Ew''$, and shear force $Q = -(Ew'')'$.

We introduce the piecewise-constant function

$$\alpha(x) = \{\alpha_j;\ x \in [x_j, x_{j+1}),\ j = 1, \ldots, n\}, \quad x_1 = 0, \quad x_{n+1} = 1, \tag{1.4}$$

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 40, No. 1, pp. 207–211, January–February, 1999. Original article submitted June 4, 1997.

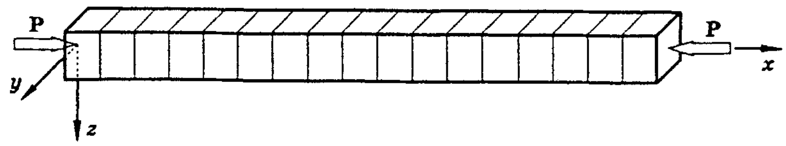

Fig. 1

which characterizes the structure of a transversely layered rod, namely, the number, dimensions, and materials of the layers forming the rod. The values of α_j belong to a discrete finite set

$$U = \{1, 2, \ldots, k\}, \tag{1.5}$$

which corresponds to the given set of materials W. Thus, all the characteristics of materials from the set W are functions of distribution $\alpha(x)$ on the interval $[0,1]$. If $\alpha_j = i$, the jth layer $[x_j, x_{j+1})$ of the rod consists of the ith material from W. The function $\alpha(x)$ is taken to be the control in the problem considered.

The problem of optimal design is formulated as follows. From the piecewise-constant functions $\alpha(x)$ (1.4) having a set of values U (1.5), we need to find a control $\alpha(x)$ that contributes minimally to the weight functional

$$F(\alpha) = \int_0^1 \rho(\alpha)\, dx \tag{1.6}$$

with the specified constraint imposed on the critical buckling load

$$P_0 - P \leqslant 0, \tag{1.7}$$

where P_0 is a specified parameter.

2. Necessary Optimality Conditions. To obtain the necessary optimality conditions for problem (1.2)–(1.7), we should construct expressions for variations in the goal functional (1.6) and constraint (1.7) through variation in the control $\alpha(x)$.

The coupling conditions at the internal boundaries between the layers of the rod enable us to introduce the following continuous phase variables on the interval $[0,1]$:

$$\mathbf{Y}(x) = (w, w', M, Q)^t \tag{2.1}$$

(t is the transpose of a matrix or vector).

Now the initial problem (1.2), (1.3) can be presented as a boundary-value problem for the desired $\mathbf{Y}(x)$:

$$\mathbf{Y}'(x) = A(\alpha, x)\mathbf{Y}(x) \tag{2.2}$$

$$y_2(0) = y_4(0) = y_1(1) = y_3(1) = 0, \tag{2.3}$$

where the nonzero coefficients a_{ij} of the matrix $A(\alpha, x)$ have the form

$$a_{12} = a_{34} = 1, \quad a_{23} = -1/E, \quad a_{43} = -P/E.$$

Let $\alpha(x)$ be the optimal control from the admissible set (1.5) which minimizes the functional (1.6) and satisfies the constraint (1.7). We consider the perturbed control $\alpha^*(x)$ [2]:

$$\alpha^*(x) = \begin{cases} g(x), & x \in D, \quad g(x) \in U, \\ \alpha(x), & x \notin D, \quad \operatorname{mes}(D) < \varepsilon \end{cases} \tag{2.4}$$

where $D \subset [0,1]$ is a set of small measure and $\varepsilon > 0$ is a small parameter. The variation of the goal functional (1.6) takes the form

$$\delta F(\alpha) = \int_D \{\rho(\alpha^*) - \rho(\alpha)\}\, dx. \tag{2.5}$$

To obtain a variation of the constraint (1.7), we express the value of the critical load P in terms of the phase variables $\mathbf{Y}(x)$ and the control $\alpha(x)$. Taking the boundary conditions (2.3) into account, from system (2.2) we obtain

$$P(\alpha, \mathbf{Y}) = \int_0^1 \frac{y_3^2}{E(\alpha)}\, dx \bigg/ \int_0^1 y_2^2\, dx. \tag{2.6}$$

Bearing in mind expression (2.6), we write the constraint (1.7) in the form

$$F_1(\alpha, \mathbf{Y}) = P_0 - P(\alpha, \mathbf{Y}) \leqslant 0. \tag{2.7}$$

Using the standard technique [2], one can obtain the principal part of the increment of the functional $F_1(\alpha, \mathbf{Y})$ (2.7):

$$\delta F_1(\alpha, \mathbf{Y}) = -\delta P(\alpha, \mathbf{Y}) = \int_D y_3^2 \left\{ \frac{1}{E(\alpha^*)} - \frac{1}{E(\alpha)} \right\} dx \bigg/ \int_0^1 y_2^2\, dx. \tag{2.8}$$

Now we construct the enhanced functional

$$J(\alpha) = F(\alpha) + \lambda\{F_1(\alpha, \mathbf{Y}) + \xi^2\} \tag{2.9}$$

(λ and ξ are the Lagrangian multiplier and the slack variable). The variation of the functional (2.9) and expressions (2.5) and (2.8) can be combined to give

$$\delta J(\alpha) = \int_D \{H(\alpha, x, \mathbf{Y}) - H(\alpha^*, x, \mathbf{Y})\}\, dx + 2\lambda\xi \cdot \delta\xi, \tag{2.10}$$

$$H(\alpha, x, \mathbf{Y}) = -\rho(\alpha) - \lambda \frac{y_3^2(x)}{E(\alpha)} \bigg/ \int_0^1 y_2^2\, dx. \tag{2.11}$$

Since the control $\alpha(x)$ is optimal (minimizing), the condition $\delta J(\alpha) \geqslant 0$ must be satisfied for any admissible function $\alpha^*(x)$ (2.4). From expression (2.10), by virtue of the arbitrariness of the variation $\delta\xi$ we obtain conditions for the supplemented nonrigidity and sign matching [3]

$$\lambda(P_0 - P(\alpha, \mathbf{Y})) = 0 \qquad (\lambda \geqslant 0), \tag{2.12}$$

and by virtue of the fact that the small-measure set D can be closely arranged almost everywhere in the interval $[0,1]$, the following condition of maximum for the Hamilton function $H(\alpha, x, \mathbf{Y})$ with respect to the argument α [2] must be satisfied for almost all values of $x \in [0,1]$:

$$H(\alpha, x, \mathbf{Y}) = \max_{\alpha^*(x) \in U} H(\alpha^*, x, \mathbf{Y}). \tag{2.13}$$

Thus, the control $\alpha(x)$ and the corresponding optimal trajectory $\mathbf{Y}(x)$ must satisfy the boundary-value problem (2.2) and (2.3), relations (1.5), (2.7), and (2.12), and the optimality condition (2.13).

3. Computational Algorithm. The method of solving the problem of optimal design consists of constructing a sequence of controls $\{\alpha(x)\}_j$ $(j = 1, 2, \ldots)$ which minimizes the goal functional (1.6). To this end we use a uniform grid $\{x_i\}$ to divide the interval $[0,1]$ into a set of intervals D_i $(i = 1, \ldots, n)$ modeling sets of small measure. We specify the initial control $\alpha(x)$ from the admissible region (1.5), (2.7). The function $\alpha(x)$ is a piecewise-constant function with constancy regions $D_i = [x_i, x_{i+1})$, where it takes values belonging to the set U (1.5). The next approximation $\alpha^*(x)$ on a certain set $D \subset \{D_i\}$ is sought in the form (2.4)

$$\alpha^*(x) = \begin{cases} \alpha_j, & x \in D, \quad \alpha_j \in U, \\ \alpha(x), & x \notin D \end{cases} \tag{3.1}$$

and is determined from the following linearized optimization problem: find, on a given set D, a perturbation α_j of the control $\alpha(x)$ that ensures the minimum of the variation $\delta F(\alpha)$ (2.5) with condition (3.1) and linearized

TABLE 1

Material	ρ	E
Spheroplastic	0.65	270
Duralumin	2.85	7100
Titanium alloy	4.6	12,000
Steel	7.8	21,000
Copper	8.93	11,200

constraint (2.7)

$$F_1(\alpha^*, \mathbf{Y} + \delta\mathbf{Y}) \approx F_1(\alpha, \mathbf{Y}) + \delta F_1(\alpha, \mathbf{Y}) \leqslant 0;$$

with allowance for (2.8), the latter can be written in the form

$$\int\limits_D \frac{y_3^2}{E(\alpha^*)}\, dx \leqslant (P(\alpha, \mathbf{Y}) - P_0) \int\limits_0^1 y_2^2\, dx + \int\limits_D \frac{y_3^2}{E(\alpha)}\, dx.$$

As a set D, one can take the elementary intervals D_i themselves and a combination of several intervals from the set $\{D_i\}$ on different parts of the rod. The variation in the control $\alpha(x)$ on several elementary intervals simultaneously can be helpful to avoid a deadlock [2], when, not being optimal, a structure cannot be refined by local variation in the control only at one of the elementary intervals.

Having constructed the new control $\alpha^*(x)$ in this fashion, we take it as the initial control and construct the next approximation. The process is considered to be finished on a given grid if the control $\alpha(x)$ does not vary for any set $D \subset \{D_i\}$. This solution is a local minimum in the problem considered.

Example. We suppose that the set W consists of five materials, whose nondimensional densities and Young's moduli are given in Table 1.

The rod is compressed by the axial force P on which the constraint $P \geqslant P_0 = 18{,}000$ is imposed. The boundary conditions (1.3) are specified at the rod's ends. The rod is divided along its length into 50 equal parts modeling the sets $\{D_i\}$.

Calculations were performed using various initial approximations, which were chosen on the basis of numerical calculations and some notions. As a result, a three-layered rod of weight $F^* = 2.814$ was obtained with critical load $P = 18{,}219$ and the layers of titanium alloy, duralumin, and spheroplastic on the intervals [0, 0.08], [0.08, 0.92], and [0.92, 1], respectively. The lightest homogeneous rod which satisfies the constraint (2.7) is that of titanium alloy with weight $F_* = 4.6$. The relative gain in weight for the optimal rod in comparison with a homogeneous rod is $(1 - F^*/F_*) \cdot 100\% = 38.8\%$.

This work was supported by the Russian Foundation for Fundamental Research (Grant No 96-01-01527).

REFERENCES

1. N. A. Alfutov, *Fundamentals of Stability Analysis for Elastic Structures* [in Russian], Mashinostroenie, Moscow (1978).
2. R. P. Fedorenko, *Approximate Solution of the Problems of Optimal Control* [in Russian], Nauka, Moscow (1978).
3. V. M. Alekseev, V. M. Tikhomirov, and S. V. Fomin, *Optimal Control* [in Russian], Nauka, Moscow (1979).